Synchronization of stochastic bistable systems by biperiodic signals

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We study the nonlinear response of a noisy bistable system to a biperiodic signal through experiments with an electronic circuit (Schmitt trigger). The signal we use is a biharmonic one, i.e., a superposition of low and high frequency harmonic components. It is shown that the mean switching frequency (MSF) of the system can be locked at both low and high frequencies. Moreover, the phenomenon of MSF locking at the lower frequency can be induced and enhanced by the higher frequency excitation. Thus high frequency bias can control synchronization at the low frequency.

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I. INTRODUCTION

The transformation of signals in a noisy environment by nonlinear systems through which they pass has attracted much attention over many years. Several nontrivial and at first sight surprising phenomena have been observed and studied. One example is the phenomenon of stochastic resonance (SR) [1], which is of great interest and importance [2–4]. SR is a cooperative nonlinear effect that is observed in a wide range of nonlinear systems simultaneously driven by noise and a weak signal. The output signal-to-noise ratio, or other measure of coherence, can be optimized for a particular noise level, so that it possesses a bell-shaped curve when plotted as a function of noise intensity. Depending on the amplitude of the signal one can distinguish linear and nonlinear regimes of system response. Theoretical studies and analog simulations of SR have shown that, for a weak signal, the phenomenon can be correctly described within the framework of linear response theory [5]. In this regime the system response does not depend on signal parameters, for example, on spectral structure, but it does depend on system nonlinearity and noise. The opposite case is a *nonlinear* regime, when the amplitude of the signal is strong enough. A variety of nonlinear effects can then be observed. The generation of higher harmonics and spectral holes has been investigated in detail [6–8]. The influence of the signal's spectral structure on the signal-to-noise enhancement was discussed in [9]. Within the nonlinear regime, synchronization of the stochastic switching dynamics by the periodic force becomes amenable to direct observation. In [10] it was shown that the mean switching frequency (MSF) of a noisy bistable system can be locked over a finite range of noise intensities, so that a region of synchronization very similar to an Arnold's tongue can be plotted in the parameter plane noise intensity vs signal amplitude. Inside this synchronization region the MSF appears to be locked by the periodic signal. In [11] a further generalization was made: it has been shown that the synchronization of switching can also be realized for a random spike train, i.e., the effect is relevant for information-carrying signals. An experimental study of the effect in a biological system was reported in [12]; the phenomenon is important for the understanding of signal processing in a neuronal system, since it specifies the conditions under which noise induces a regime of complete (optimal) information transmission and simultaneously it is observed in a wide range of noise amplitudes.

The effects described above were observed for signals with a single dominant time scale [27]. In real systems, however, it is often the case that there may be several superimposed signals with different time scales. Such superposition can result from the mixing of a signal with jamming interference in a radio transmission, or in the activity of two distinct areas of a neural network. In the case of a weak signal, the presence of additional components does not change the transformation of each individual component. However, for a larger signal, interactions between its components and the system nonlinearity will play a role. Indeed, a number of phenomena have been observed in the specific case of two-component signals, e.g., an experimental realization of noise-enhanced heterodyning in a bistable optical system [13]. Another interesting effect is the so-called ghost resonance [14] observed in bistable systems forced by noise and a signal that consists of two close harmonic components. In the latter case, system nonlinearity produces the SR effect for a beat frequency. The possible relevance of ghost resonance for signal processing in auditory systems has been widely discussed [14,15]. In [16] the noise was replaced by a high-frequency harmonic excitation with a much higher frequency than the signal. An SR-like behavior of the spectral gain factor was observed, an effect that was named vibrational resonance, and a maximum of the gain factor is observed by transition from sub- to suprathreshold excitation. The presence of noise induces gain degradation in vibrational resonance [17–19].

In the present paper we investigate switching synchronization of a noisy bistable system due to a biperiodic signal consisting of low and high frequency components. Thus we introduce two different time scales into the signal. There are several motivations for the study. First, as it is well known, classical SR is a low-frequency phenomenon, i.e., it is at its most pronounced for low frequency signals. The questions that arise here are whether an additional high frequency bias can control synchronization at the low frequency or, conversely, how the low frequency bias changes the system response to the high frequency component. Another motivation

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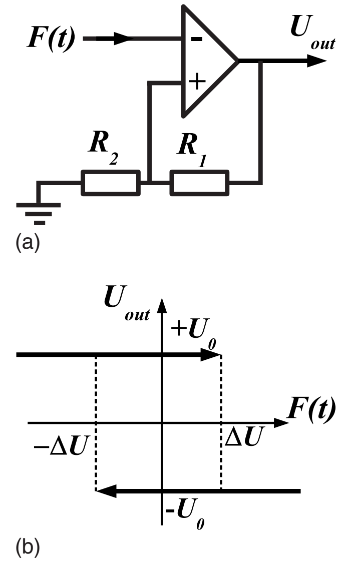

FIG. 1. A circuit diagram of Schmitt trigger (a) and its (idealized) input-output hysteresis characteristic (b).

relates to possible applications of SR for the transmission and enhancement of information-carrying signals. The simplest model of such a signal is a superposition of harmonic signals with different amplitudes and frequencies. We consider the case of two harmonic components with frequencies that differ by a factor of ten. To investigate the problem, we choose a simple bistable system in the form of an electrical circuit (the Schmitt trigger). The paper is organized as follows. In Sec. II we introduce the model and specify the experimental setup. The measures that we apply to quantify the synchronization phenomena are discussed in Sec. III. The results of the measurements are discussed theoretically in Sec. IV and the conclusions drawn are summarized in Sec. V.

II. EXPERIMENTAL ARRANGEMENTS

To analyze the switching process we chose an archetypal system demonstrating pure two-state dynamics: the Schmitt trigger, for which a schematic circuit and its hysteresis characteristics are presented in Fig. 1. An ideal Schmitt trigger circuit driven by external force $F(t)$ obeys the equation

$$U_{out} = \mathrm{sgn}[\Delta U - F(t)], \tag{1}$$

where "sgn" is the sign function and the threshold values ΔU are defined by resistors R_1 and R_2:

$$\Delta U = \frac{R_1}{R_1 + R_2} U_{out}, \tag{2}$$

where the output voltage $U_{out} = \pm U_0$ is equal to either the positive or negative voltage $\pm U_0$ of the power source. The value of the output voltage is controlled by external force $F(t)$, i.e., if $F(t) > |\Delta U|$ the trigger is in its upper state $+U_0$, if $F(t) < -|\Delta U|$, then it is in the state $-U_0$. Equation (1) describes the circuit correctly provided that the relaxation rate of the trigger is the fastest scale and that it is much shorter than either the signal time scale or the correlation time of the noise.

A Gaussian noise generator, used as a noise source $\xi(t)$, produces colored noise with a Lorentzian spectrum, characterized by a cutoff frequency of 100 kHz. The noise undergoes additional filtering by a circuit element (mixer), so that the tail of the spectrum at the trigger input differs from the Lorentzian. The noise energy is characterized by its root-mean-square amplitude σ as measured by an rms voltmeter. Two independent harmonic signal generators are used to provide the two-frequency force. The circuit response is digitized by an analog-to-digital converter (ADC) and passed to a computer for analysis. The ADC was a DAS-1600 with a maximum sampling rate of 10^5 points per second in the regime of input buffering. The relaxation rate of the trigger was 1 MHz.

From such a two-frequency signal $F_s(t)$ one can single out its low-frequency (LF) and high-frequency (HF) components as

$$F_s(t) = A_L \sin(2\pi f_L t) + A_H \sin(2\pi f_H t), \tag{3}$$

where f_L and f_H are the frequencies, and A_L and A_H are the amplitudes, of the LF and HF parts, respectively. It is intuitively evident that the two time scales in such a regular force must lead to a mutual influence of each component on the other. Let us fix the threshold level $|\Delta U| = \Delta^* = 470$ mV and consider the response of the Schmitt trigger forced by the two-frequency signal $F_s(t)$ and colored noise $\xi(t)$:

$$U_{out} = \mathrm{sgn}[\Delta U - F_s(t) - \xi(t)]. \tag{4}$$

For convenience of representation of the results, the amplitudes of the regular signal A_L and A_H, as well as noise amplitude σ, are normalized by the value of the same threshold level Δ^*. For the theoretical discussion, all values of frequencies used in experiments were normalized by 1 kHz.

In our experiments the regular signal is subthreshold, $A_L + A_H < \Delta U$, and all the switchings between states are therefore induced by noise. So the experiment differs from the arrangement used for vibrational resonance and corresponds to the usual conditions used for SR and MSF locking studies.

We set the signal frequencies arbitrarily at $f_L = 100$ Hz and $f_H = 1006$ Hz and analyze the influence of each component of the composite two-frequency (biharmonic) signal on stochastic synchronization. Note that although the ratio between the selected frequencies f_L and f_H is rational, and that the signal is consequently periodic, this fact is not important in itself here since the period of signal is much larger than the periods of both the LF component and the measurement time. Moreover, if we choose a signal period equal to one period of the LF component, e.g., $f_H = n f_L$ ($n \geq 10$ is integer), all the effects to be discussed below will be observed and additional fine structure will appear in the phase distribution. In other words, since the frequencies f_L and f_H differ by one order of magnitude, the observed effects do not depend on whether the ratio between the frequencies is rational or irrational.

III. STOCHASTIC SYNCHRONIZATION

The effect of MSF locking by an external harmonic signal was reported in [10] and it was shown that this phenomenon can be considered as an example of stochastic synchronization [11,12,20,21]. In contrast with synchronization in deterministic systems, stochastic synchronization considers interaction between a number of processes one or several of which is or are random. For example, when the MSF is locked, the harmonic signal controls the stochastic switchings of the bistable system and, within a certain interval of noise intensities, the MSF coincides with the frequency of the periodic signal. For a symmetrical bistable system, the locking of the MSF is defined by the probability that a random process crosses a boundary (barrier) between the states for a specific range of signal phase. In the presence of a harmonic signal this boundary is periodically nonstationary and stochastic synchronization is observed under the condition that the first passage time to the boundary $+|\Delta U|+A\sin\varphi$ for $\varphi\in(\pi,2\pi)$ tends to infinity, whereas the passage to the boundary $+|\Delta U|+A\sin\varphi$ for $\varphi\in(0,\pi)$ occurs during a finite time that is less than half the period of the signal. That is, the phase intervals in which a switching does or does not occur are strongly emphasized. The switching for the values of signal phase $\varphi\in(0,\pi)$ for which the probability of transition to a state is larger than the probability transition to another state can be determined as in-phase switching. Otherwise, one can speak of out-of-phase switching. Note that, in contrast to the classical usage of the terms "in-phase" and "out-of-phase," a phase *interval* is meant here. Within the approximation that the switching probability is equal to unity for switching in-phase, and to zero for switching out-of-phase, the situation corresponds to stochastic synchronization. Thus the effect of MSF locking can be considered as the phenomenon of phase and frequency stochastic synchronization.

The conditions of MSF locking formulated above are of course subject in practice to finite observation time. If an ideal source of Gaussian noise is applied, satisfaction of the strict condition that the switching probability is equal to zero (in a theoretical limit of infinite observation time) at a given phase is impossible because of the nonvanishing tail of the noise distribution. But, since the tail is exponential, and the observation time is in reality finite, satisfaction of the synchronization condition depends on the parameters of system and forcing. In practice, one can compare the switching probability with a small constant value instead of with zero.

The switching process in a bistable system can be characterized by the MSF and the evolution of the residence time distribution $p(\tau)$ and phase (or cycle) distribution $p(\varphi)$ [22]. The phase distribution $p(\varphi)$ estimates the probability of switching at the given phase φ of the harmonic signal. In the case considered one can separately introduce the phase distributions for the LF $p(\varphi_L)$ and HF $p(\varphi_H)$ harmonic components. The residence time $p(\tau)$ is the time interval τ between two successive switchings. The MSF $\langle f\rangle$ is defined by the residence time distribution as follows:

$$\langle f\rangle = \left[\lim_{t'\to\infty}\frac{2}{t'}\int_0^{t'}\tau p(\tau)d\tau\right]^{-1}. \tag{5}$$

In the experiments, a thousand switching events were collected to evaluate the MSF and the corresponding distribution for each set of parameters. The trigger output signal and the signal from the lower-frequency harmonic generator were simultaneously recorded to determine the switching phase. The moments of switching and values of the switching phase were calculated using a linear interpolation of points in the time series.

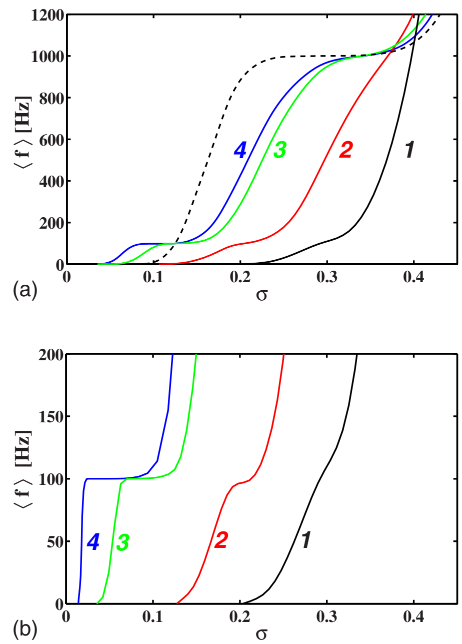

FIG. 2. (Color online) Experimental results obtained from the circuit model shown in Fig. 1. (a) The mean switching frequency (MSF) $\langle f\rangle$ is plotted as a function of noise amplitude σ (dimensionless units) for different combinations of amplitudes of the two-frequency signal with $\Delta U=1$. The full curves correspond to $A_L=0.2$ and $A_H=0$ (line 1), $A_H=0.43$ (line 2), $A_H=0.64$ (line 3), $A_H=0.7$ (line 4); the dashed curve corresponds to $A_L=0$, $A_H=0.7$. (b) Plots of the MSF $\langle f\rangle$ as a function of σ for different values of the threshold: $\Delta U=1$ (line 1), $\Delta U=0.9$ (line 2), $\Delta U=0.34$ (line 3), $\Delta U=0.25$ (line 4). The HF component is absent $A_H=0$ and the amplitude of LF component $A_L=0.2$.

IV. EXPERIMENTAL RESULTS AND THEORETICAL CONSIDERATIONS

First, the evolution of the system response was measured as the amplitude of the HF component was increased, keeping the amplitude of the LF component fixed at $A_L=0.2$. Figure 2(a) plots the mean switching frequency as a function of the input noise amplitude σ for different values of the HF amplitude A_H. In the absence of the HF part of the signal, the MSF increases monotonically (dashed line) in the range of low frequency f_L. As the amplitude of the HF component increases, switchings occur for smaller values of noise amplitude, and flat regions corresponding to synchronization can be observed. The latter expand with increasing strength of the HF component. Thus the regime of MSF locking at low frequency has been realized by application of the higher frequency external excitation. To compare the influences of the HF component and variations in the threshold value for harmonic signal the plots of $\langle f\rangle$ vs σ for different thresholds are presented in Fig. 2(b). Note that the curve 1 is the same for both figures (a) and (b), but it is shown for different ordinate scales. As it is seen, the increase in amplitude of the HF component is qualitatively equivalent to a decrease in the threshold value: the curves shift to the direction of smaller noise amplitude and the synchronization regime is observed. Below we provide additional support of this equivalence.

For A_H=0.7 [curve 4 in Fig. 2(a)], the MSF is constant in two regions of noise amplitude. In the first one, the MSF coincides with the low frequency f_L and, in the second one, with the high frequency f_H. Let us consider this situation.

Without LF bias, the locking of the MSF at f_H has been realized [dashed curve in Fig. 2(a)]. In the presence of the LF component with A=0.2 a region of synchronization at frequency f_L arises and the synchronization area for frequency f_H decreases. Thus the locking of the mean switching frequency is realized at both frequencies f_L and f_H.

To understand how the HF component influences the response of the trigger to the LF component, we can average over the fast time scale $T_H=1/f_H$. The threshold-crossing rate for a Gaussian process in the limit of adiabatic driving takes the following form [23] (see also [24–26])[28]:

$$r_{\pm}(t) = \frac{1}{2\pi}\sqrt{-K''(0)}\exp\left(-\frac{[\Delta U - F_s(t)]^2}{2\sigma^2}\right); \qquad (6)$$

here $K''(0)$ is the second derivative of the autocorrelation function of the noise (in the current experiment, $|K''(0)|$ =0.02 ms [29]), the positive sign corresponds to the threshold value $+|\Delta U|$ and switching to the state $+U_0$, whereas the negative sign corresponds to $-|\Delta U|$ and switching to the state $-U_0$. To evaluate the crossing rate averaged over T_H, we introduce the rate with an effective threshold ΔU_e:

$$r_{+} = \frac{1}{2\pi}\sqrt{-K''(0)}\exp\left(-\frac{(\Delta U_e)^2}{2\sigma^2}\right) = \frac{1}{2\pi}\sqrt{-K''(0)}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(-\frac{(|\Delta U| - A_H\cos\varphi)^2}{2\sigma^2}\right)d\varphi. \qquad (7)$$

Then using a Gaussian approximation of the integrand in the form $\exp(-\varphi^2/2\sigma_\varphi^2+C)$, where σ_φ and C are unknown constants defined for $\varphi=0$ and $\varphi=\pi/2$, one can obtain the expression for the effective threshold ΔU_e,

$$(\Delta U_e)^2 = (|\Delta U - A_H|)^2 - 2\sigma^2\ln\left[\frac{\sqrt{\pi}}{2s_1}\operatorname{erf}(s_1)\right], \qquad (8)$$

where

$$s_1 = \frac{\sqrt{2A_H(2|\Delta U| - A_H)}}{\sigma}.$$

Consequently, the HF excitation effectively decreases the value of the threshold for the LF component. In term of effective threshold the time-dependent crossing rate for the LF component is

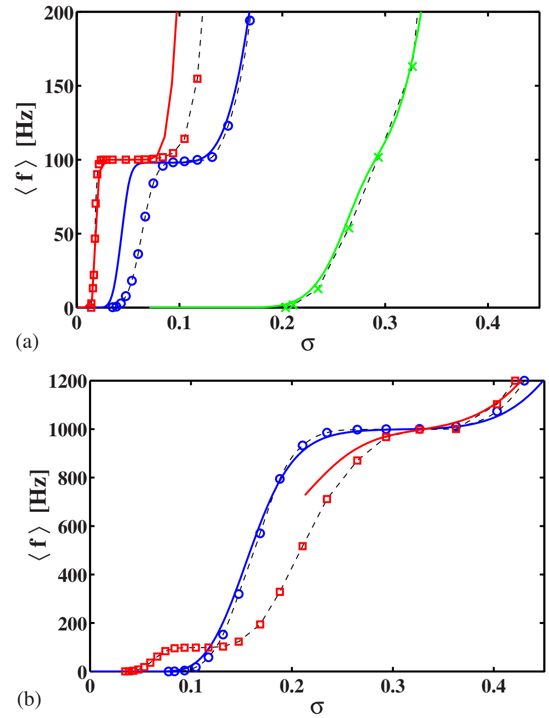

FIG. 3. (Color online) Comparison of experimentally measured (data points) and calculated (full curves) MSFs $\langle f\rangle$ as functions of noise amplitude σ (dimensionless units). The dashed lines connecting data points are guides to the eye. (a) The influence of the HF component on MSF locking at the low-frequency f_L. The data points × correspond to the parameter set ΔU=1, A_L=0.2, and A_H=0; ○ to ΔU=1, A_L=0.2, and A_H=0.7; □ to ΔU=0.25, A_L=0.2, and A_H=0. (b) The influence of the LF component on MSF locking on the high-frequency f_H. The data points ○ correspond to the parameter set: ΔU=1, A_L=0, and A_H=0.7; □ to ΔU=1, A_L=0.2, and A_H=0.7.

$$r_{\pm}(t) = \frac{1}{2\pi}\sqrt{K''(0)}e^{(-[\pm\Delta U_e - A_L\sin(2\pi f_L t)]^2/2\sigma^2)}. \qquad (9)$$

The MSF can be obtained by using the time-periodic solution of the master (rate) equation for the population $n_{\pm}(t)$ of the states $+U_0$ and $-U_0$ [21,22]:

$$\langle f\rangle = f_L\int_0^{T_L} r_{-}(t)n_{+}(t)dt. \qquad (10)$$

The rate equation has the form [21,22]

$$\dot{n}_{+} = -r_{-}n_{+} + r_{+}n_{-},$$

$$\dot{n}_{-} = -r_{+}n_{-} + r_{-}n_{+} \qquad (11)$$

and is solved numerically starting, for example, with identical initial conditions: $n_{+}(0)=n_{-}(0)=0.5$; then, after a transient time, the time-periodic solution is used in Eq. (10).

The theoretical and experimental results presented in Figs. 2 and 3(a) clearly show that increasing the amplitude of the HF component leads to a decrease in the threshold value for

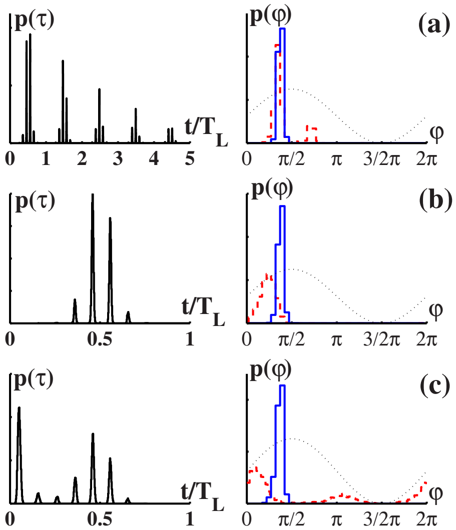

FIG. 4. (Color online) The distributions $p(\tau)$ and $p(\varphi)$ in the region of synchronization at f_L (a) $\langle f\rangle < f_L$, (b) $\langle f\rangle = f_L$, (c) $\langle f\rangle > f_L$. Time is normalized by the period of the LF component $T_L = 1/f_L$ in the plots of $p(\tau)$.

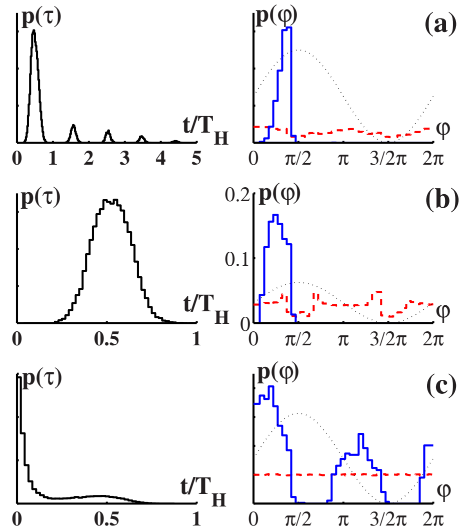

FIG. 5. (Color online) The distributions $p(\tau)$ and $p(\varphi)$ in the region of synchronization at f_H (a) $\langle f\rangle < f_H$, (b) $\langle f\rangle = f_H$, (c) $\langle f\rangle > f_H$. Time is normalized by the period of the HF component $T_H = 1/f_H$ in the plots of $p(\tau)$.

the LF component and hence induces MSF locking on frequency f_L. The agreement between experiment and theory can be considered satisfactory.

The experimental results also show clearly that, in turn, LF bias influences synchronization at f_H. Without the LF component there is a wide region of MSF locking on f_H [dashed line in Fig. 2(a)]; the LF component induces a significant decrease of the region (curve 4). The presence of LF part can be considered to produce a slow trend in the signal. In this case the synchronization conditions for the HF component should be modified by replacing ΔU with $\Delta U - A_L \cos(2\pi f_L t)$. The periodic change of the threshold can be approximated by a step function $\Delta U - 2A_L/\pi\,\mathrm{sgn}[\cos(2\pi f_L t)]$, i.e., for one half period the threshold is $\Delta U - 2A_L/\pi$ and for the other half it is $\Delta U + 2A_L/\pi$. Then we can calculate two MSFs (10) for the two threshold values separately, and estimate the resultant MSF as the mean of these two. This approach is valid for $\langle f\rangle \gg f_L$. The curve to which it leads fitted the experimental results quite well [Fig. 3(b)]. Thus the presence of a slow trend leads to a degradation of stochastic synchronization.

We now consider the distributions of residence times $p(\tau)$ and phases $p(\varphi)$ in the synchronization regime with both components of the forcing frequency $F_s(t)$. For this purpose we plot distributions $p(\tau)$ and $p(\varphi)$ before, at the moment of, and after synchronization at frequencies f_L and f_H (Figs. 4 and 5). The solid (blue) line corresponds to the HF component and the dashed (red) line to the LF component in the plots of $p(\varphi)$. The dependences of the amplitude of a harmonic signal on phase are shown by the dotted lines.

If the mean switching frequency is smaller than the low frequency f_L, the distribution $p(\tau)$ consists of two sharply pronounced time scales. One scale is defined by the period T_L of the LF component, the other is defined by T_H. As shown in Fig. 4(a) the structure of exponentially decreasing peaks centered at $T_L/2 + nT_L$ $(n=1,2\ldots)$ is modulated by the frequency f_H. The distribution $p(\varphi)$ shows that the switchings are in-phase for both components, and the phases of the LF and HF components lie within the interval from 0 to π. In the region of synchronization at f_L a plot of the residence time distribution demonstrates several peaks [Fig. 4(b)], their envelope corresponding to the synchronization by a harmonic signal. For both components, the switchings are in-phase and, for the LF component, the region of possible switching phase is wider than for the HF part.

The distributions $p(\tau)$ and $p(\varphi)$ after the breakdown of synchronization at frequency f_L are shown in Fig. 4(c). Out-of-phase switchings for the LF component appear but, for the HF component, the distribution $p(\varphi)$ has nonzero values only in a narrow region of phase located in the interval $\varphi \in [0,\pi]$. The peaks in $p(\tau)$ corresponding to the out-of-phase switchings with respect to the LF component are defined by the period of the HF part.

If the MSF is larger than the low frequency f_L, but smaller than f_H, then the influence of the LF component is not seen in either of the distributions $p(\tau)$ and $p(\varphi)$ [Fig. 5(a)]. The peak locations in the distribution of residence times are defined by the HF component alone: switchings occur in-phase but the switching probability for any value of the low-frequency phase is nonzero.

Figure 5(b) shows the plots of distributions for the regime of synchronization at f_H. The switchings occur every half period of the HF component. The phase distribution for the LF part is flat.

When out-of-phase switchings with respect to the HF component appear, the synchronization at frequency f_H breaks down and a new short-time peak appears in the residence time distribution [Fig. 5(c)]. For the HF component the distribution $p(\varphi)$ includes phase intervals with zero switching probability and two peaks, the distance between them being equal to π. If the noise intensity is increased further, the peak centered at the half period time is destroyed and the switching probability becomes nonzero for all phases of the HF component, i.e., the switching process is then defined by noise only (not shown).

For both components of the two-frequency signal, the evolution of the distribution $p(\varphi)$ up to the point of synchronization breakdown can be characterized by the dependence of the average phase on noise amplitude. Obviously, the magnitude of the average phase coincides with the phase for which the switching probability is maximal. At the moment of switching, the average phase is close to $\pi/2$ and, with increase of the noise amplitude, the value of the average phase approaches zero. Consequently, the switching moment is defined not only by the value of signal amplitude but also by the sign of the derivative of the regular signal.

V. CONCLUSIONS

In summary, our investigation of stochastic synchronization in a bistable system forced by a biharmonic signal allows us to conclude that the phenomenon of synchronization depends on relationships between the parameters of the signal's components. Synchronization at the low frequency was realized and enhanced by introduction of the higher frequency excitation. Correspondingly, the region of MSF locking at the low frequency depends on the amplitude of the HF bias. Locking of the MSF by both components of the regular force can be realized, the locking of MSF by the LF component being induced by the high-frequency excitation. The effect of synchronization is defined by the time scales of both components. On the other hand, for MSF locking by the HF component, the presence of a slow trend (the LF component) leads to a degradation of stochastic synchronization. These results are important not only for physics but perhaps also for biology as a possible mechanism of information transmission by sensory neurons.

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[27] In the case of a spike train one can introduce a unique instantaneous frequency, i.e., only one "time varying" time scale.

[28] We have to note that in [26] an exact crossing rate for a double linear filtered noise has been derived. This rate includes a multiplicative correction term. We do not take into account this correction here since for the considered range of parameters the correction is close to the unit value.

[29] This value was obtained by fitting the experimental curve of crossing rate MSF vs σ by expression (6) for $F_s=0$.